The mathematics of Luca Pacioli: appropriation, not plagiarism

Luca Pacioli (c. 1447–1517) conference

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Albrecht Heeffer

Research fellow
Centre for History of Science
Ghent University, Belgium

Agenda

- Claims of plagiarism
- Context: six phases towards the algebra textbook
- Pacioli at the start of algebraic theory
- Pacioli's contributions to mathematics

Claims of plagiarism (1)

- The third part of *De divina proportione* (1509) is translated from Piero della Francesca's *De quinque corporibus regularibus*
- Giorgio Vasari, *Lives of the most eminent painters sculptors & architects*, vol 3., p. 22
  - Now Maestro Luca dal Borgo, a friar of S. Francis, who wrote about the regular geometrical bodies, was his pupil; and when Piero, after having written many books, grew old and finally died, the said Maestro Luca, claiming the authorship of these books, had them printed as his own, for they had fallen into his hands after the death of Piero.
- Status: Pacioli defended by Arrighi (1970), Biggiogere (1960), Frajese (1967), Jayawardene (1974, 1976), Ricci (1940)

Claims of plagiarism (2)

- The part of perspective in *De divina proportione (1509)* is based on Piero della Francesca's *De prospectiva pingendi* (c. 1474)
- Status: claims of plagiarism are relative
  - Acknowledged by Pacioli in the *De divina proportione* and the *Summa*

Claims of plagiarism (3)

- The *Queste e el libro che tracta di mercatantie et usanze de paese* (Summa 1494) is based on Giorgio Chiarini 1481
- Girolamo Mancini. L'opera "de corporibus regularis" di Pietro Franceschi detto della Francesca, usurpata da Fra' Luca Pacioli. Rome: Tip. della R. accademia dei Lincei,1915.
- Status: no plagiarism
  - Tariff tables now considered common property of 15th century merchant culture
  - Similar tables existed before Chiarini

Claims of plagiarism (4)

- The geometrical part of the *Summa* (1494) is based on an abbaco mansucript
- Ettore Picutti, "Sui plagi matematici di frate Luca Pacioli", *Scienze*, 256, 72–79, 1989.
  - "all the 'geometria' of the Summa, from the beginning to page 59v. (118 pages numbered in folios), is the transcription of the first 241 folios of the Codex Palatino 577".
- Status: plagiarism, no translation, no acknowledgement, but the problems appear in many other 15th century manuscripts

Claims of plagiarism (5)

- Many parts on arithmetic and algebra in the *Summa* (1494) are copied from abbaco manuscripts
- Franci, Rafaella and Laura Toti Rigatelli (1985) "Towards a history of algebra from Leonardo of Pisa to Luca Pacioli", *Janus*, 72 (1-3), pp. 17-82.
  - "Unmerited fame"
  - "Comparing the later [Summa] with large handwritten treatises [abbaco ms..] shall give many surprises"
- Status: appropriation, no plagiarism

Phases in development of a mathematics textbook

1. The medieval tradition (800–1202)
2. Abacus manuscripts (1307–1494)
3. The beginning of algebraic theory (1494–1539)
   - Extracting general principles
4. Algebra as a model of demonstration (1545–1637)
5. From problems to propositions (1608–1643)
6. Axiomatic theory (1657–1830)

Heeffer, Albrecht. 2012. "The Genesis of the Algebra Textbook: From Pacioli to Euler", *Almagest*, 3 (1), pp. 26–61.

1. The medieval tradition

- Problems as vehicles for rote-learning
- Example: Alquin's *Propositiones ad acuendos juvenes*
  - Two men were leading oxen along a road, and one said to the other: "Give me two oxen, and I'll have as many as you have." Then the other said: "Now you give me two oxen, and I'll have double the number you have." How many oxen were there, and how many did each have?
- Rhetoric of master and student
  - Declamation of a problem and asking for an answer
  - Rhyme and cadence essential in memorization
- Also in Hindu algebra (Brāhmagupta, Mahāvira,..)

2. The abbaco tradition

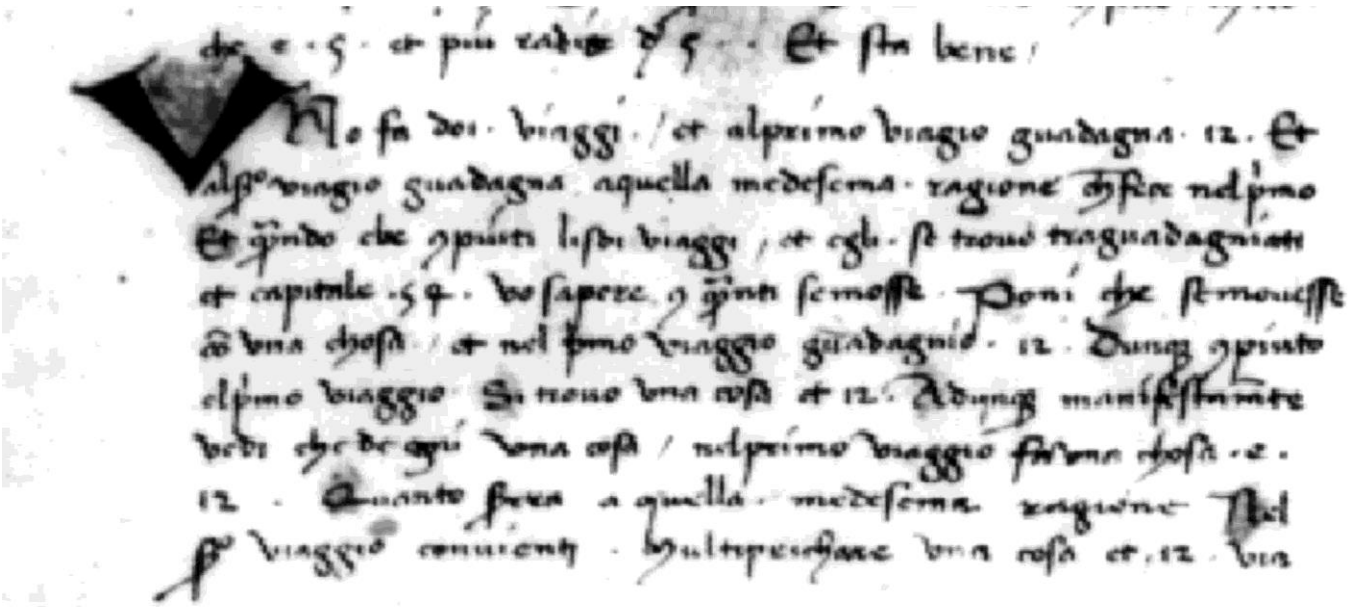
- Problems as algebraic practice
- Continuous development before Fibonaci (1202) till after Pacioli (1494)
- Learning by problem solving
- Knowledge disseminated between master and apprentice (often in family relations)
- Texts present the rhetorical reformulation of a problem using a *cosa*.
- Elegance of the solution depends on a clever choice of the unknown(s) (Antonio de' Mazzighi, c. 1380):

Abbaco algebra follows a rigid rhetorical structure

1. Problem enunciation
2. Choice of the rhetorical unknown
3. Manipulation of polynomials
4. Construction of an ‘equation’ solvable by a standard rule
5. Root extraction
6. Numerical test

Example 1

Earliest known abbacus manuscript on algebra



Jacopo da Firenze, ms. Vat. Lat. 4862, f. 39^{v} (1307)

1. Enunciation of the problem

- Someone makes two business trips. On the first he makes a profit of 12. On the second he wins in the same proportion and when he ends his trip he found himself with 54. I want to know with how much he started with.
- Uno fa doi viaggi, et al primo viagio guadagna 12. Et al secondo viagio guadagna a quella medesema ragione che fece nel primo. Et quando che conpiuti li soi viaggi et egli se trovò tra guadagniati et capitale 54. Vo' sapere con quanti se mosse.

2. Choice of the unknown

Uses a (modified of combined) unknown quantity of the problem as the rhetorical unknown

- Pose that one begins with one cosa.
- Poni che se movesse con una cosa.

3. Manipulating polynomials

Nel primo viagio guadangniò 12, dunque, compiuto il primo viagio si truova 1 cosa e 12, adunque manifestamente apare che d'ongni una cosa faegli 1 cosa e 12 nel primo viagio. Adunque, se ogni una cosa fae una cosa e 12, quanto far`a una cosa e 12. Convienti multiprichare una cosa e 12 via una cosa e 12 e partire in una cosa. [f. 30v]. Una cosa e 12 via una cosa e 12 fanno uno cienso e 24 cose e 144 numeri, il quale si vuole partire per una cosa e deve venire 54. E perci`o multipricha 54 via una cosa, fanno 54 cose, le quali s'aguagliano a uno cienso e 24 cose e 144 numeri. Ristora ciaschuna parte, cio[è] di chavare 24 cose di ciaschuna parte.

Manipulating polynomials

- And on the first trip he wins 12. Then completing his first trip he finds 1 cosa and 12.
- It is then also manifest that for each cosa one obtains 1 cosa and 12 on the first trip. How much does this become in the same proportion after the second trip?
- It is appropriate to multiply one cosa and 12 with one cosa and 12 which makes one censo and 24 cosa and 144 numbers, which will become 54.
- And therefore multiply 54 with one cosa. Makes 54 cose, which is equal with one censo and 24 cose and 144 numbers.
- Restore each part, therefore subtract 24 cose from each part.

$$x+12$$

$$\frac{x+12}{x} : \frac{54}{x+12}$$

$$(x+12)(x+12)$$

$$x^2+24x+144$$

$$54x$$

$$x^2+24x+144=54x$$

$$x^2+144=30x$$

4. Constructing an ‘equation’

- You will have that 30 cose are equal to one censo and 144 numbers.

- Averai che 30 cose sono iguali a uno cienso e 144 numeri.

$$30x = x^2 + 144$$

Arabic type V ‘equation’

5. Root extraction

Applying a cannonical recipe:

- Divide in one censo, which becomes itself. Then take half of the cose, which is 15. Multiply by itself which makes 225, subtract the numbers which are 144, leaves 81. Find its [square] root which is 9. Subtract it from half of the cose, which is 15. Leaves 6, and so much is the value of the cosa.
- Parti in uno censo, vene quello medesemo. Dimezza le cose, remanghono 15. Multipricha per se medesemo, fanno 225. Traine li numeri, che sonno 144, resta 81. Trova la sua radice, che è 9. Trailo del dimezzamento dele cose, cioè de 15. Resta 6, et cotanto vale la chosa.

$$bx = ax^2 + c$$

$$bx = x^2 + c$$

$$x = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

$$x = \frac{30}{2} - \sqrt{\left(\frac{30}{2}\right)^2 - 144} = 6$$

6. The test

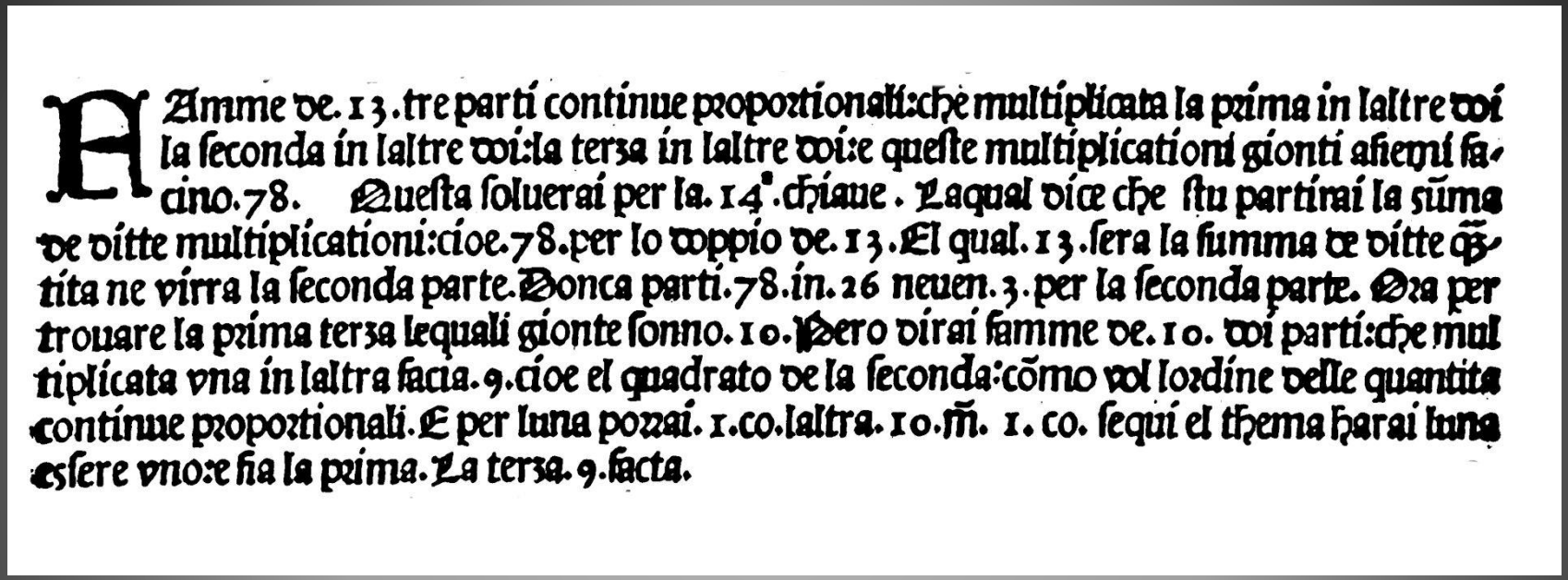
- And if you want to prove this, do as such. You say that on the first trip one wins 12 and with the 6 one started with, one has 18. So that on the first trip one finds 18. Therefore say as such, of every 6 I make 18; what makes 18 in the same proportion? Multiply 18 with 18, makes 324. Divide by 6, this becomes 54, and it is good.

Et se la voi provare, fa così. Tu di' che nel primo viaggio guadagnio 12 et con 6 se mosse a 18. Siché nel primo viaggio se trovò 18. E peró di' così, se de 6 io fo 18, que farò de 18 a quella medesema ragione? Multipricha 18 via 18. Fa 324. Parti in 6, che ne vene 54, et sta bene..

Pacioli constructing algebraic theory

- Problems for generating algebraic theory
- Extracting general principles from practice
- Transformation of rhetoric of problem solving
  - Solved problems become theorems
- Case 1
  - Pacioli *Summa* 1494 (numbers in continuous proportion)
  - Taken from Antonio de' Mazzinghi (c.1390)

Pacioli 1494 f. 91r



AZamme de. 13. tre parti continue proportionali: che multiplicata la prima in laltre doi la seconda in laltre doi: la terza in laltre doi: e queste multiplicationi gionti ahiemi facino. 78. Questa soluerai per la. 14°. chiaue. Laqual dice che stu partirai la sũma de ditte multiplicationi: cioe. 78. per lo doppio de. 13. El qual. 13. sera la summa de ditte q̃titta ne virra la seconda parte. Donca parti. 78. in. 26 neuen. 3. per la seconda parte. Ora per trouare la prima terza lequali gionte sonno. 10. Dero dirai famme de. 10. doi parti: che multiplicata vna in laltra facia. 9. cioe el quadrato de la seconda: cõmo vol lordine delle quantita continue proportionali. E per luna pozzai. 1. co. laltra. 10. m̃. 1. co. sequi el thema harai luna essere vno: e fia la prima. La terza. 9. facta.

Make three parts of 13 in continuous proportion so that the first multiplied with [the sum of] the other two, the second part multiplied with the [sum of the] other two, the third part multiplied with the [the sum of the] other two, and these sums added together makes 78.

de' Mazzinghi, Trattato di Fioretti

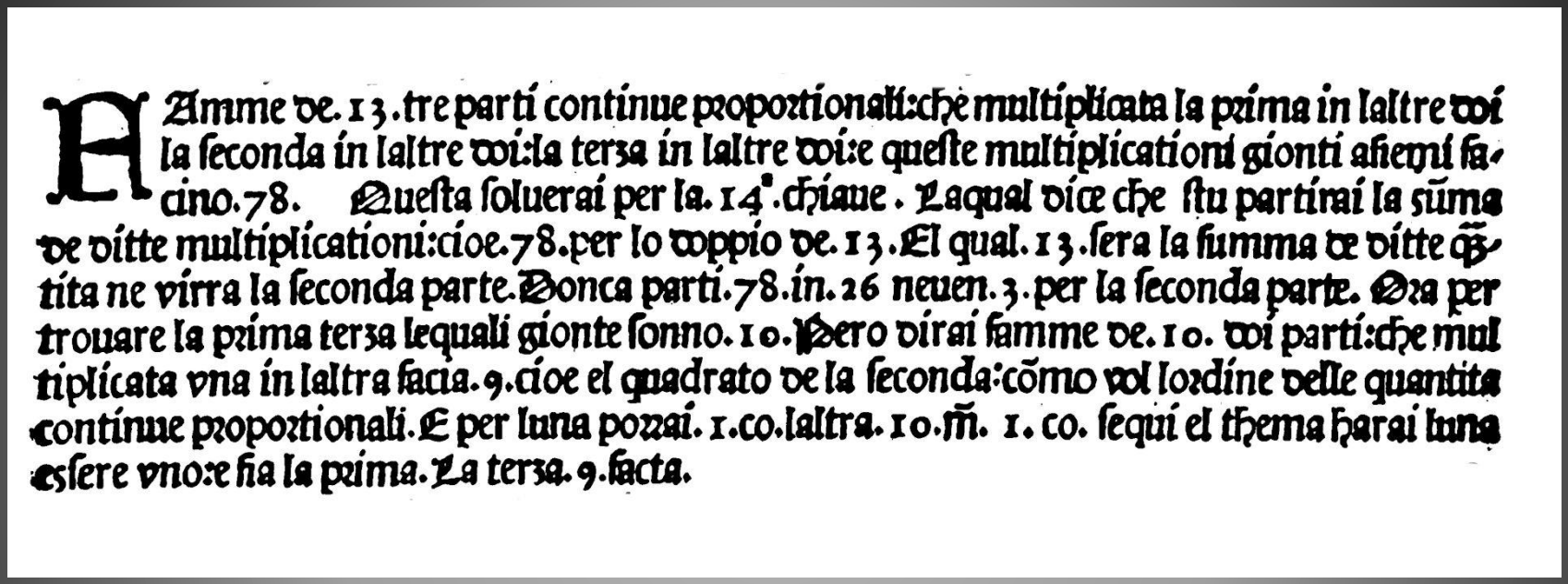
Arrighi 1967, p. 15: "Fa' di 19, 3 parti nella proportionalità chontinua che, multiplichato la prima chontro all'altre 2 e lla sechonda parte multiplichato all'altre 2 e lla terza parte multiplichante all'altre 2, e quelle 3 somme agunte insieme faccino 228. Adimandasi qualj sono le dette parti".

Make three parts of 19 in continuous proportion so that the first multiplied with [the sum of] the other two, the second part multiplied with the [sum of the] other two, the third part multiplied with the [the sum of the] other two, and these sums added together makes 228. Asked is what are the parts.

de' Mazzinghi, Trattato di Fioretti

- The problem in modern symbolism:
  - $\frac{x}{y} = \frac{y}{z}, x + y + z = a, x(y + z) + y(x + z) + z(x + y) = b$
- Expanding the product and summing the parts:
  - $2xy + 2xz + 2yz = 228$
- But as $y^2 = xz$ this can be expressed as:
  - $2xy + 2y^2 + 2yz = 228$ or $2y(x + y + z) = 228$
- With the sum being 19, $2y$ thus equals 12 or y=6
- The problem reduces to dividing 13 into two parts with 6 as the middle term, leading to
  - $x^2 + 36 = 13x$

Pacioli 1494 f. 91r



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This can be solved using the fourteenth key. Which says that you have to divide the sum of these multiplications, thus 78, by the double of 13. And this 13 is the sum of these quantities, which will give you the second part. Thus divide 78 by 26 gives 3 for the second part.

Pacioli constructing algebraic theory

- Problem

$$\frac{x}{y} = \frac{y}{z}$$

$$x + y + z = a$$

$$x(y+z) + y(x+z) + z(x+y) = b$$

- Formulation of general key 14
  - "On three quantities in continuous proportion, when multiplying each with the sum of the other two and adding these products together. Then divide this by double the sum of these three quantities and this always gives the second quantity".

$$y = \frac{x(y+z) + y(x+z) + z(x+y)}{2(x+y+z)} = \frac{b}{2a}$$

Pacioli constructing algebraic theory

- Summa, dist. 6, treatise 6, art 10-12
  - Three numbers in GP: 15 keys
  - Four numbers in GP: 8 keys
- Summa, dist. 6, treatise 6, art 14
  - Three and four numbers in GP: 29 of 35 problems
- Most problems taken from *Trattato di Fioretti*
  - same problem, same values
  - same problem, different values
  - variations on problems

Algebraic theory before Pacioli

- Magl. Cl. XI. 119, Problem RAA303 (c. 1417)
  - Enunciation: Fammi di 10 tali 2 parte che multipricata l'una contro all'altra faccia 16.
  - Solution: Noi sappiamo che è 2 e 8 ma facciamo questo leggieri per intendere le più forti. Farai cos: pogniamo che quello numero fosse una cosa...
  - Test: Esse la vuoi provare dirai radicie di 9 sie 3 agiugni sopra 5 sono 8...
  - Rule: In questa regola potremmo mostrare più leggieremento ma non farebbe regola della cosa e fa così: dirai il 1/2 di 10 sie 5. Multiprica 5 in sé fa 25. Trai 16 di 25 resta 9. E rispondi e di' l'uno è 5 26 più radicie di 9 e l'altro e 5 meno radicie di 9.
- A. Heeffer 2009. "Text production reproduction and appropriation within the abbaco tradition: a case study" *Sources and Commentaries in Exact Sciences*, 9, pp. 211–256.

Algebraic theory after Pacioli

Extended by Cardano (1539) chap. 42 and 51

- Three numbers in continuous proportion

$$y = \frac{x(y+z)+y(x+z)+z(x+y)}{2(x+y+z)}$$

- Four numbers in continuous proportion

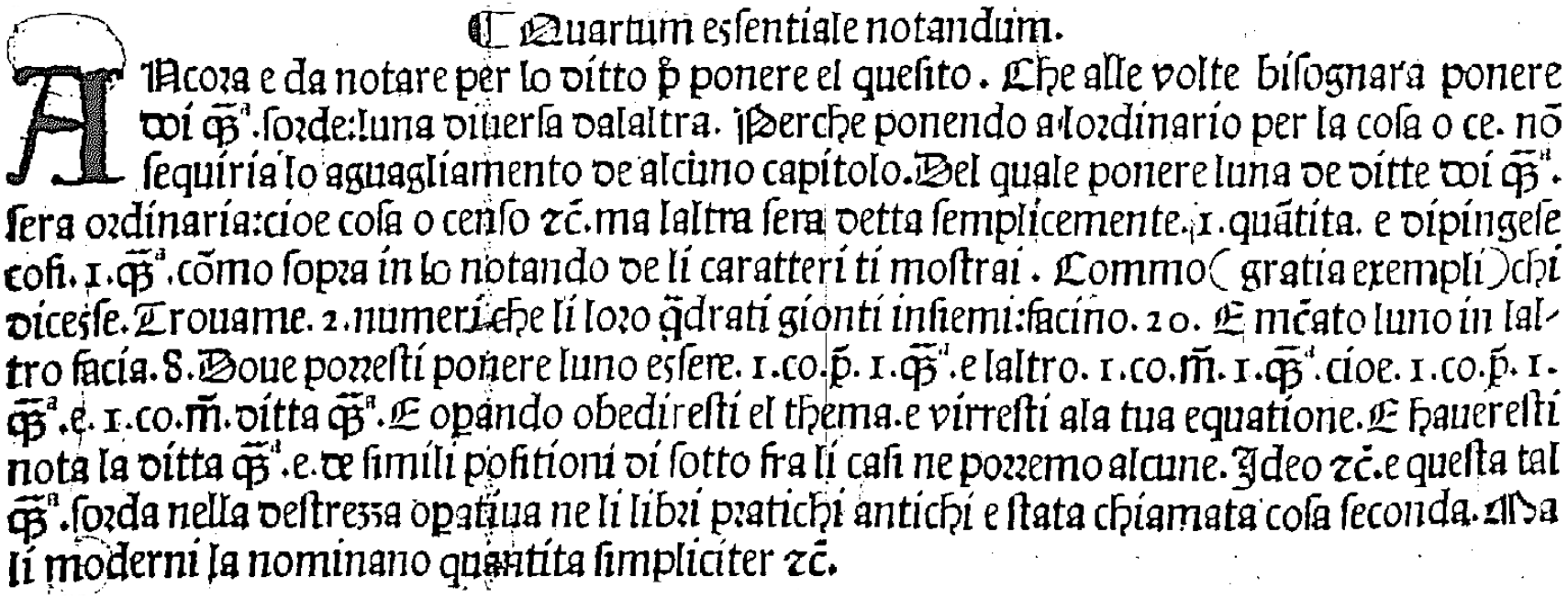
$$\left(\frac{x+y+z+u}{x+u}\right) = \frac{x+z}{x+z-y} + \frac{y+u}{y+u-z}$$

Conclusion about plagiarism

- Pacioli borrowed a lot from the abbaco tradition
  - Antonio de' Mazzinghi
  - Piero della Francesca
- Pacioli contributed to the teaching of algebra
  - Extracting theory from algebraic practice
  - Generalizing problems on numbers in GP
- The *Summa* is an important bridge between the closed manuscript tradition and the mathematics books of the 16th century

Case 2: The second unknown

- Method explained in the *Summa*, dist. 8, treat. 6, f. 148^{v}
- cosa and quantita



Quartum essentiale notandum.

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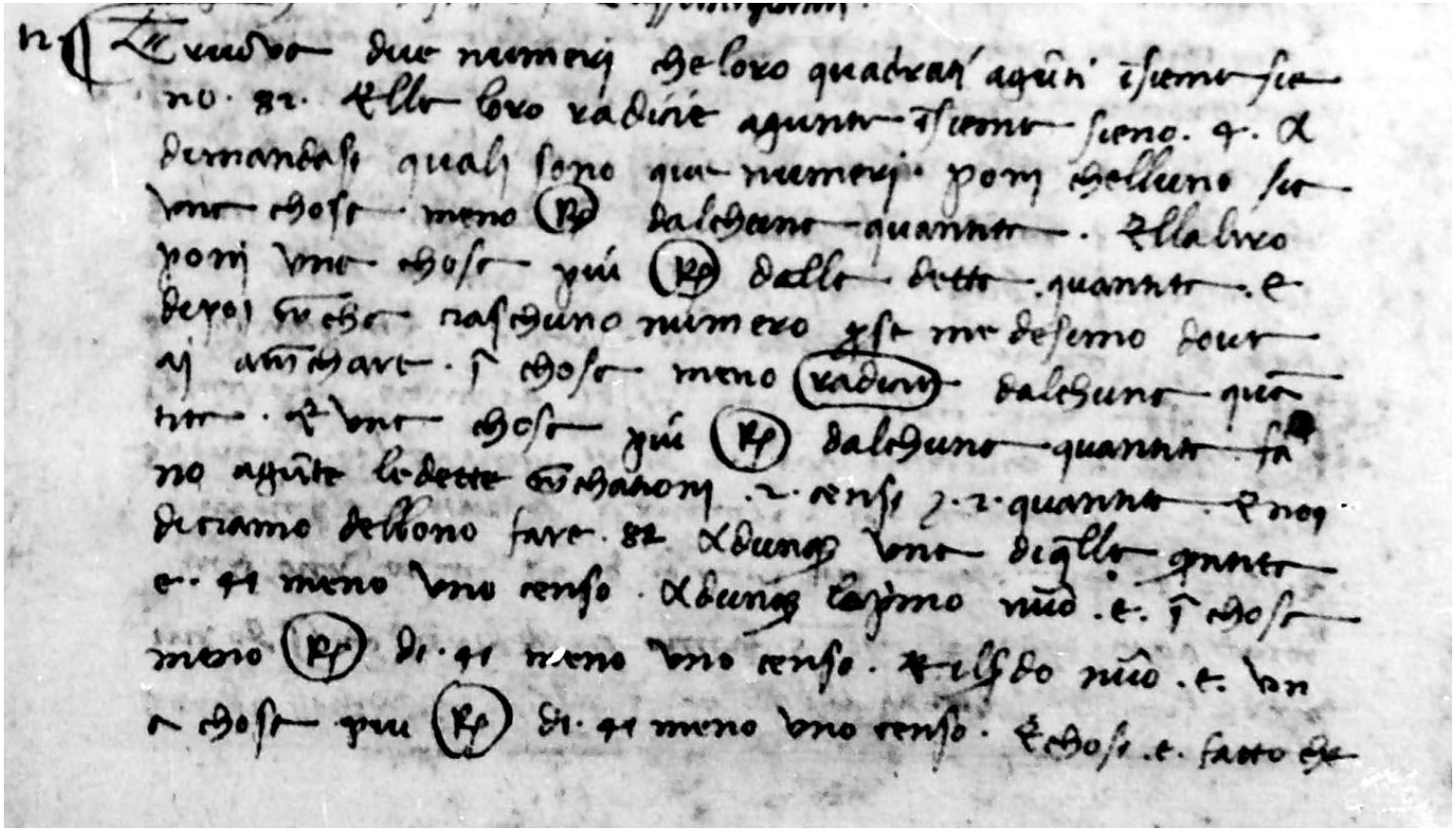
Case 2: The second unknown

- Antonio de' Mazzighi (c. 1380) was the first (after Fibonacci) to use the second unknown for solving problems
  - cosa and quantità
- Used in *Trattato di Fioretti*
- Also used in Palatino 573

De' Mazzinghi (Pal. 573 BNCF, f. 486r)



Solution with two unknowns

$$\begin{cases} a^2+b^2=82 \\ \sqrt{a}+\sqrt{b}=4 \end{cases}$$

$$\begin{cases} a=x-\sqrt{y} \\ b=x+\sqrt{y} \end{cases}$$

$$x^2-2x\sqrt{y}+y+x^2+2x\sqrt{y}+y=82$$

$$2x^2+2y=82,\ y=41-x^2$$

$$\begin{cases} a=x-\sqrt{41-x^2} \\ b=x+\sqrt{41-x^2} \end{cases}$$

$$a+b+2\sqrt{ab}=16$$

$$2x+2\sqrt{x^2-\left(41-x^2\right)}=16$$

$$2x+\sqrt{8x^2-164}=16$$

$$16-2x=\sqrt{8x^2-164}$$

$$256+4x^2-64x=8x^2-164$$

$$4x^2+64x=420$$

$$x^2+16x=105$$

$$x=5$$

$$y=41-25=16$$

$$a=1, b=9$$

Pacioli appropriating de' Mazzinghi

- Find two numbers with the sum of their squares equal to 20 and their product equal to 8
- Pacioli uses $x - y$ and $x + y$ for the numbers
  - (Pacioli 1494, f. 148^{v}): "Dove ponesti ponere l'uno essere 1.co p. 1q^{a} e l'altro 1 co m. & q^{a}"

$$(x^2+2xy+y^2)+(x^2-2xy+y^2)=20,$$
$$(x-y)(x+y)=x^2-2xy-y^2=8$$

Pacioli appropriating de' Mazzinghi

- Pacioli 1494, f. 192^{r}: "E per via de queste quantita sorda quali li antichi chiamavano cose seconde: si solvano moltissime forte rasoni chi ben le manegeia in li aguaglimenti ma te conven sempre fare che la quantita resti sola da un lato e da l'altro sia che vole m*eno* o piu che non sa caso tutto sera valuta dela quantita e reca sempre tutto a uno quantita".
- And by way of this *quantita sorda*, which the ancients called the second unknown: so they can solve much harder problems, handling the equations well, which they always fit in such a way that the unknown appears only at one side [of the equation] and the other more or less, so that they not know in all cases the values of all the unknowns and therefore they always bring everything to one unknown.

Case 3: use in linear problems

Anonymous, BNCF, Fond. prin. V.152, c. 1390

- Men buying a horse (ox) of unknown price

Tre ànno danari e vogliono chonperare una ocha e niuno di loro non à tanti danari che per sé solo la possa chonperare; or dice il primo agli altri due: se ciaschuno di voi mi desse il 1/3 de' suo danari i' chonprerei l'ocha. Dice il sechondo agli altri due: se voi mi date il 1/3 più 4 de' vostri danari i' chonperò l'ocha. Dice il terzo agli altri due: se voi mi date il 1/4 meno 5 de' vostri danari i' chon però l'ocha. Poi agiunsono insieme i danari ch'eglino aveva no tra tutti e tre e posonglì sopra la valuta del'ocha ella somma farà 176, adimandasi quanti danari aveva chatuno per sé e che valeva l'ocha (f. 177r)

$$a+\frac{1}{3}(b+c)=d$$

$$b+\frac{1}{4}(a+c)+4=d$$

$$c+\frac{1}{4}(a+b)-5=d$$

$$a+b+c+d=176$$

First use in linear problems

- Uses *chosa* an *ocha* as unknowns
- Uses algebra to the point of two expressions in two unknowns
- Finds value using double false position
- Other example:

$$a = x, d = y$$

$$\frac{1}{3}(b+c) = y - x$$

$$b + c = 3y - 3x$$

$$a + b + c = 3y - 2x$$

$$a + b + c + d = 4y - 2x = 176$$

$$\begin{cases} 7y = 13x + 4 \\ 4y = 2x + 176 \end{cases}$$

$$\begin{cases} 73x - 7y = 664 \\ 3x + 7y = 552 \end{cases}$$

Maestro Benedetto

- Not in the *Trattato di praticha d'arismetrica* (Siena)
- *Trattato d'Abacho* (c. 1460, 18 copies)
  - Six problems solved with second unknown
    - "Men find a purse" and "men buy a horse"
  - Uses *quantità* and *chavallo* or *borsa*
  - Sum of the shares as the first unknown
  - Resolves indeterminacy by way of the two unknowns:

$$29y = 17x$$

Piero della Francesca

- Trattato d‘abaco (c. 1480)
  - Florence, Biblioteca Medicea-Laurenziana, Ashburn 280 (359* 291*) [3]^{r}–127^{v}
- Three problems with the second unknown
  - Linear problem: uses a triangle superscript for the second unknown (c. 39v)
  - Linear problem: *cavallo* and *cosa* (c. 40r)
  - Problem GP: *cosa* and *quantitá* (c. 125v)

Chuquet *Triparty*, 1484

Used in linear problems in the *Appendice*

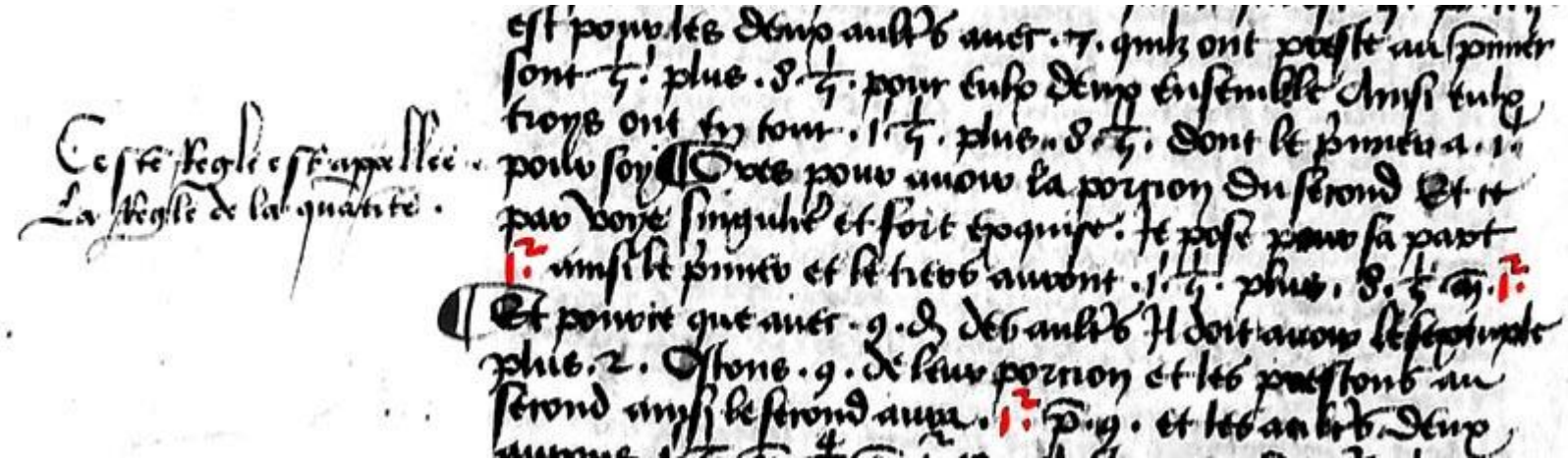
- Problems 71, 75, 77, 78, 79 (Marre, 1881)

$$a+7=5(b+c-7)+1$$
$$b+9=6(a+c-9)+2$$
$$c+11=7(a+b-11)+3$$

- Same problem in Fibonacci and Barthelemy
  - Proto-algebraic rule

Chuquet *Triparty*, 1484

- Chuquet uses 1^2 for the second unknown (f. 196^v)
- De la Roche: “Ceste regle est appellee La Regle de la quantite”

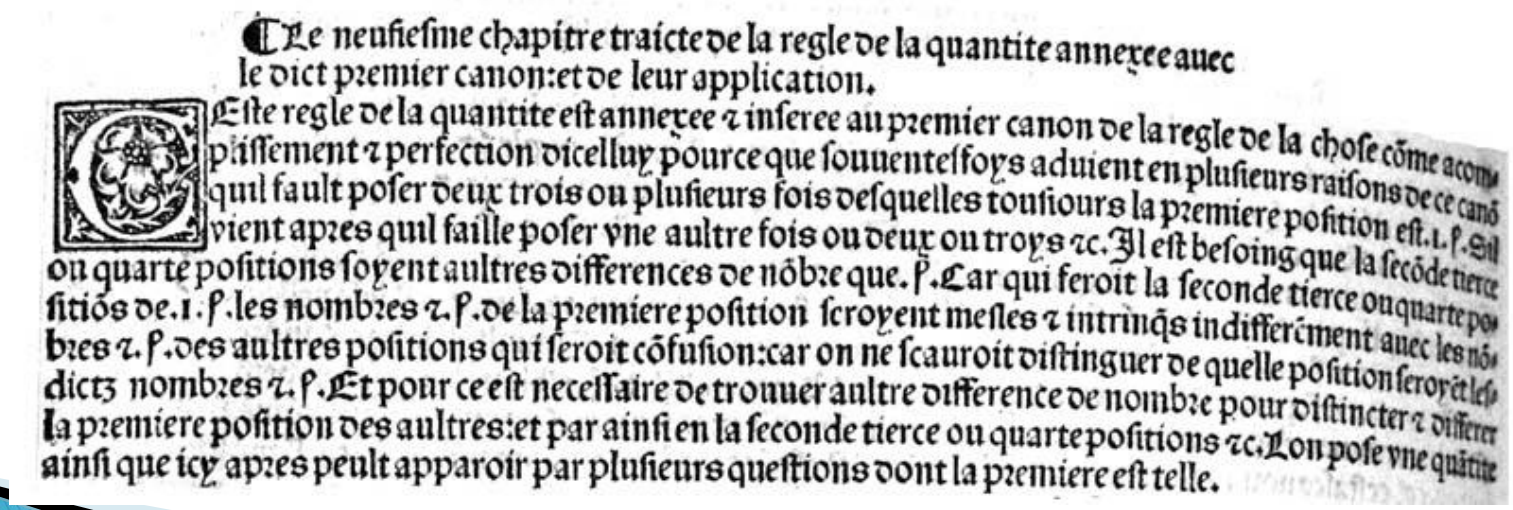


de la Roche, *Larismetique*, 1520

- Accused of plagiarism by Marre (1880, 1881)
  - Does not reproduce the method for the same problem
  - Reorganizes the text and improves notation
- The first to give a name and description of the method
  - "La regle de la quantite"

de la Roche, *Larismetique*, 1520

- Removes ambiguity by using ρ and *Qtite* for *x* and *y*
  - "It is therefore necessary that the second, third or fourth position should be a number different from ρ. Because when the numbers for the second, third and fourth positions are the same and indistinguishable from the numbers for ρ, or the other positions, this would lead to confusion"



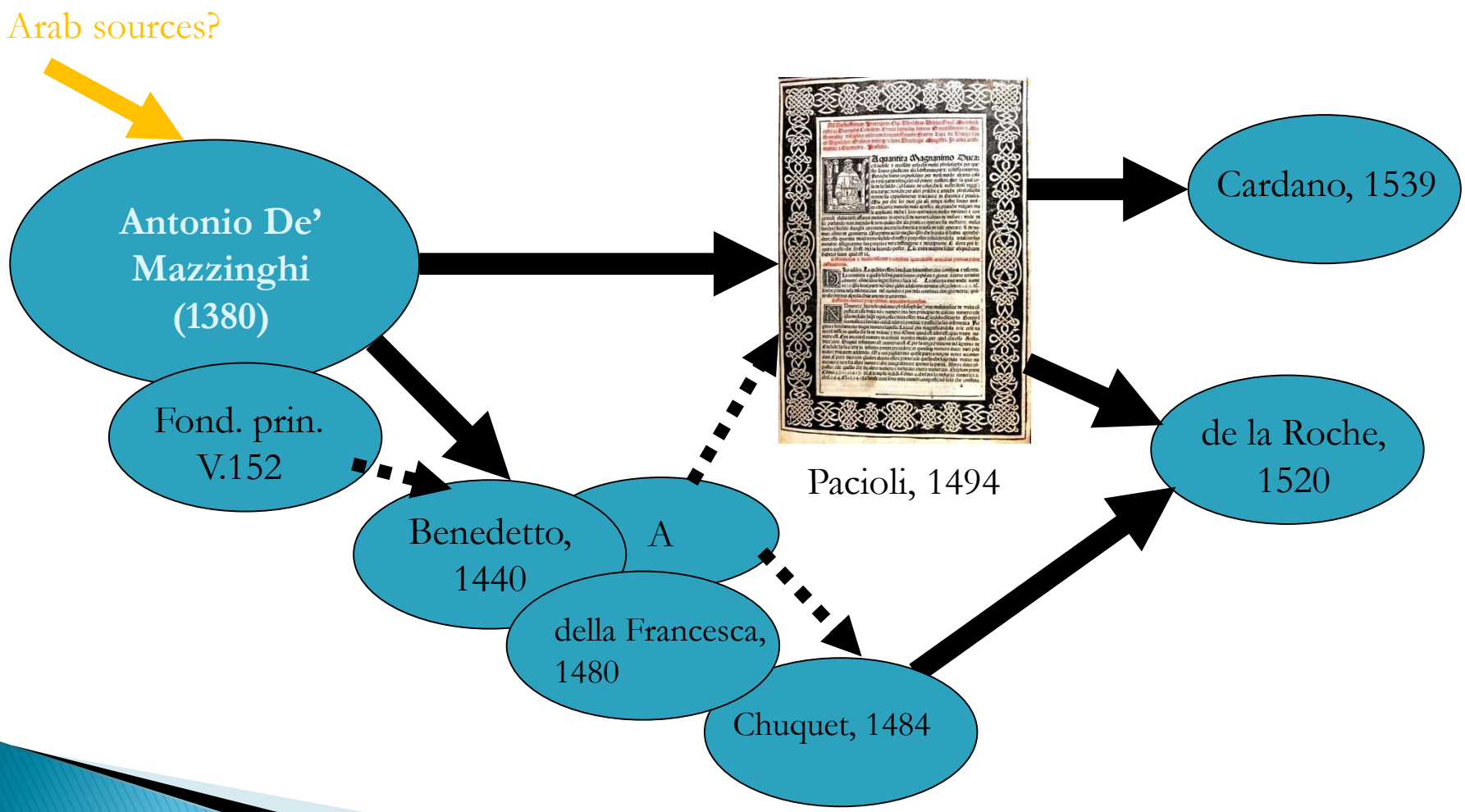
¶ Le neufiesme chapitre traicte de la regle de la quantite annexee auec le dict premier canon: et de leur application.

Este regle de la quantite est annexee & inseree au premier canon de la regle de la chose cõme accomplissement & perfection dicelluy pource que souuentesfoys aduient en plusieurs raisons de ce canõ quil fault poser deux trois ou plusieurs fois desquelles tousiours la premiere position est. 1. ρ. Sil vient apres quil faille poser vne aultre fois ou deux ou troys &c. Il est besoing que la secõde tierce ou quarte positions soyent aultres differences de nõbre que. ρ. Car qui feroit la seconde tierce ou quarte positiõs de. 1. ρ. les nombres &. ρ. de la premiere position seroyent mesles & intrinqs indifferẽment auec les nõbres &. ρ. des aultres positions qui feroit cõfusion: car on ne scauroit distinguer de quelle position seroyẽt les dictz nombres &. ρ. Et pour ce est necessaire de trouuer aultre difference de nombre pour distincter & differer la premiere position des aultres: et par ainsi en la seconde tierce ou quarte positions &c. Lon pose vne quãtite ainsi que icy apres peult apparoir par plusieurs questions dont la premiere est telle.

Luca Pacioli, *Summa,* 1494

- Uses second unknown in three ways:
  1. Reproduces M° Antonio's problems on number in continuous proportion (pt. 1, distinction 6, treatise 6)
  2. General explanation of "*quantita sorda* ne li libri pratichi antichi e stata chiamata cosa seconda" (pt. 1, distinction 9, treatise 6)
  3. Uses *cavalo* as second unknown (pt. 1, distinction 9, treatise 8) (without realizing so ?)

Use of the second unknown < 1540



Luca Pacioli, *Summa*, 1494 (2)

- Probably based on ms *A*
- Same method as Chuquet
  - First unknown *a*
  - Second unknown *b*
  - Sum of three expressed in *x*
- Same problem appears in Catalan writings
  - Barcelona Ms. 71 c. 1500 and Ventallor (1521)
  - Do not adopt the method of two unknowns

$$a+\frac{1}{2}(b+c)=50$$

$$b+\frac{1}{3}(a+c)=50$$

$$c+\frac{1}{4}(a+b)=50$$

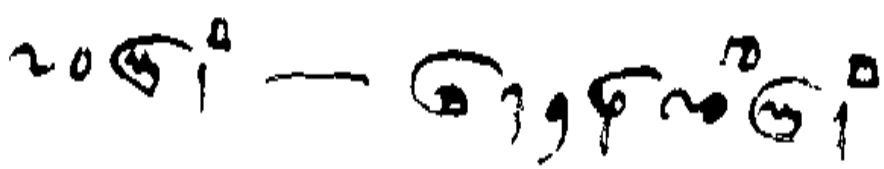
One last achievement of Pacioli

- By the 1470's there was a consistent system for algebraic notation in one unknown using an "equation sign"
  - Regiomontanus (c 1463)
  - Piero della Francesa
  - Luca Pacioli (1478)
- This notation system was not fully transferred to print

Pacioli in manuscript

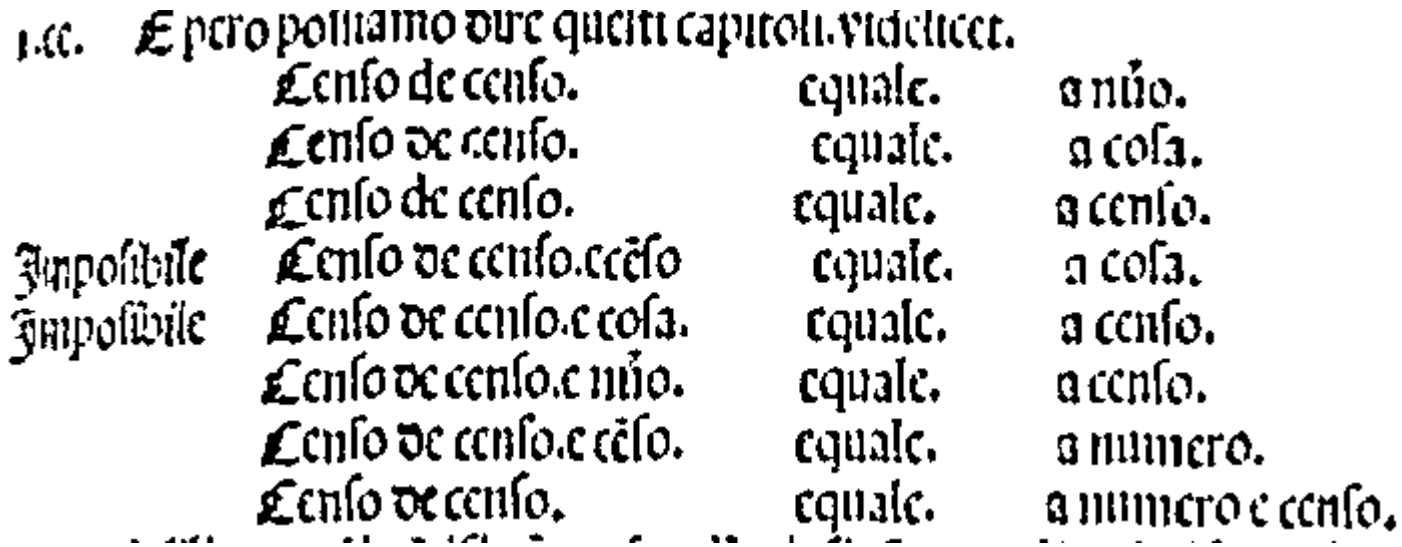
- 1478, Vat. Lat. 3129, about 600 pages
  - A. Heeffer, "Algebraic partitioning problems from Luca Pacioli's Perugia manuscript (Vat. Lat. 3129)" in *Sources and Commentaries in Exact Sciences*, (2010), 11, pp. 3–52.
- Consistent symbolism throughout the text
- example: fol. 236^{v}
- symbols for equations using $+$, $-$, $=$, x, x^2

$$20 - x^2 = -39 + 20x - x^2$$



Pacioli in print (1494)

- “equations” in full words (*Summa* f. 149r)

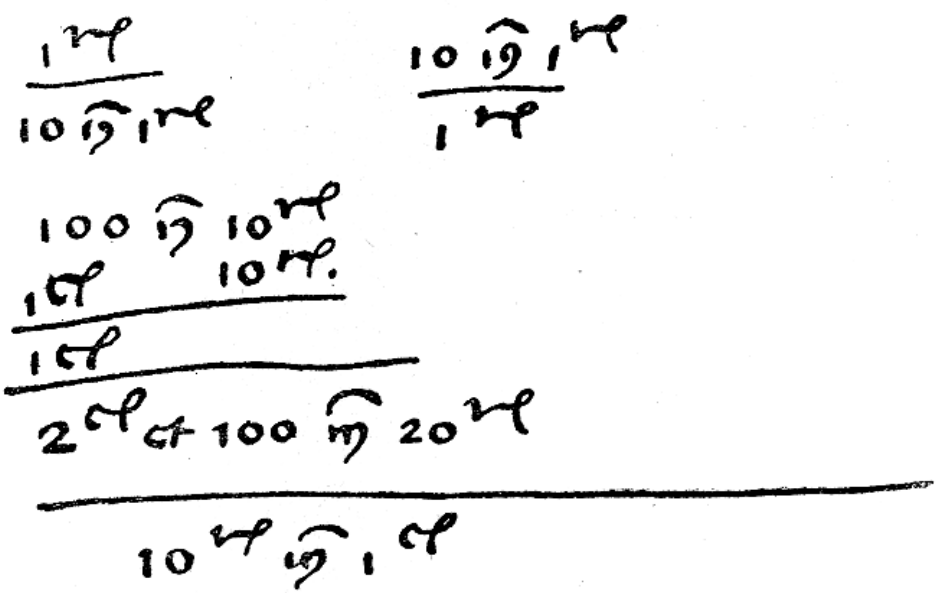


1.cc. E pero poniamo due quesiti capitoli. videlicet.

	Censo de censo.	equale.	a nũo.
	Censo de censo.	equale.	a cosa.
	Censo de censo.	equale.	a censo.
Impossibile	Censo de censo.cc̃so	equale.	a cosa.
Impossibile	Censo de censo.c cosa.	equale.	a censo.
	Censo de censo.c nũo.	equale.	a censo.
	Censo de censo.c c̃so.	equale.	a numero.
	Censo de censo.	equale.	a numero c censo.

- s
  - co. ce. m. p. eq. Rx
- no other mathematical ligatures or symbols

Regiomontanus (c 1460)

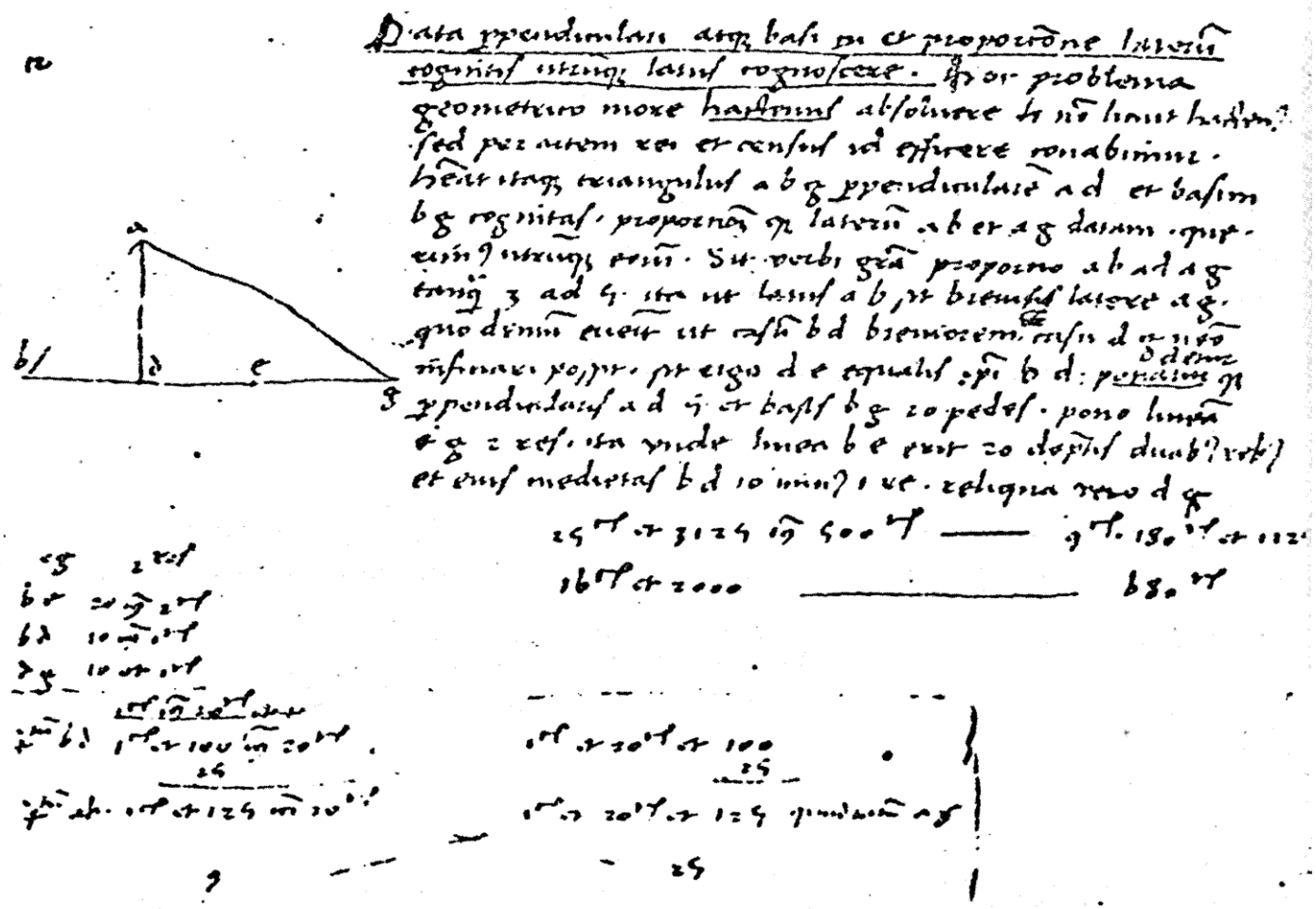


Nürnberg Cent. V 56c, f. 23

$$\frac{x}{10-x}, \frac{10-x}{x}$$

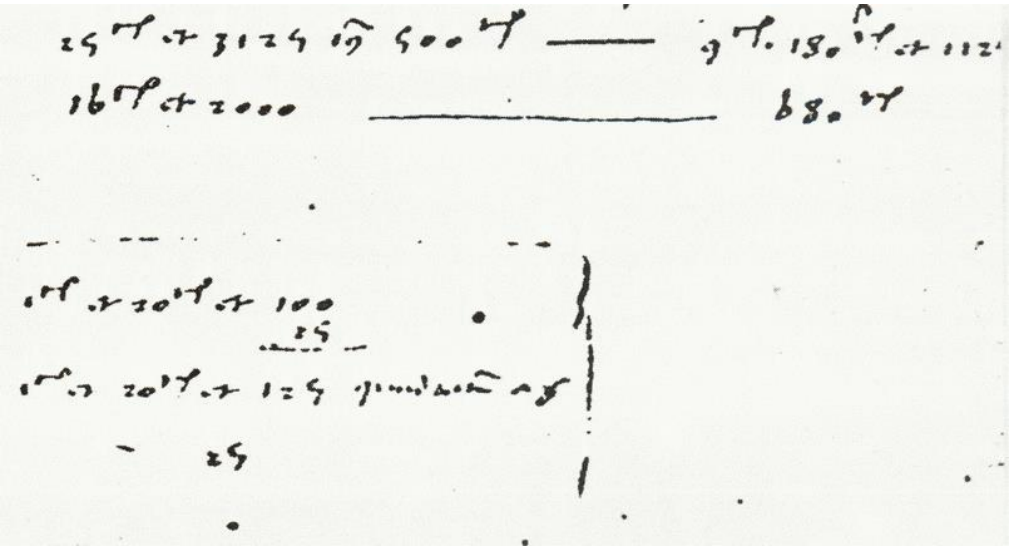
$$\begin{array}{l} 100-10x \\ \underline{x^2-10x} \\ \quad x^2 \\ \hline 2x^2+100-20x \\ \hline 10x-x^2 \end{array}$$

Regiomontanus ms. Triangulis



Moscou MS. 541, f. 40v, book II, prop. XII (written c. 1463)

Regiomontanus ms. *De Triangulis*



$$25x^2 + 3125 - 500x = 9x^2 + 180x + 1125$$

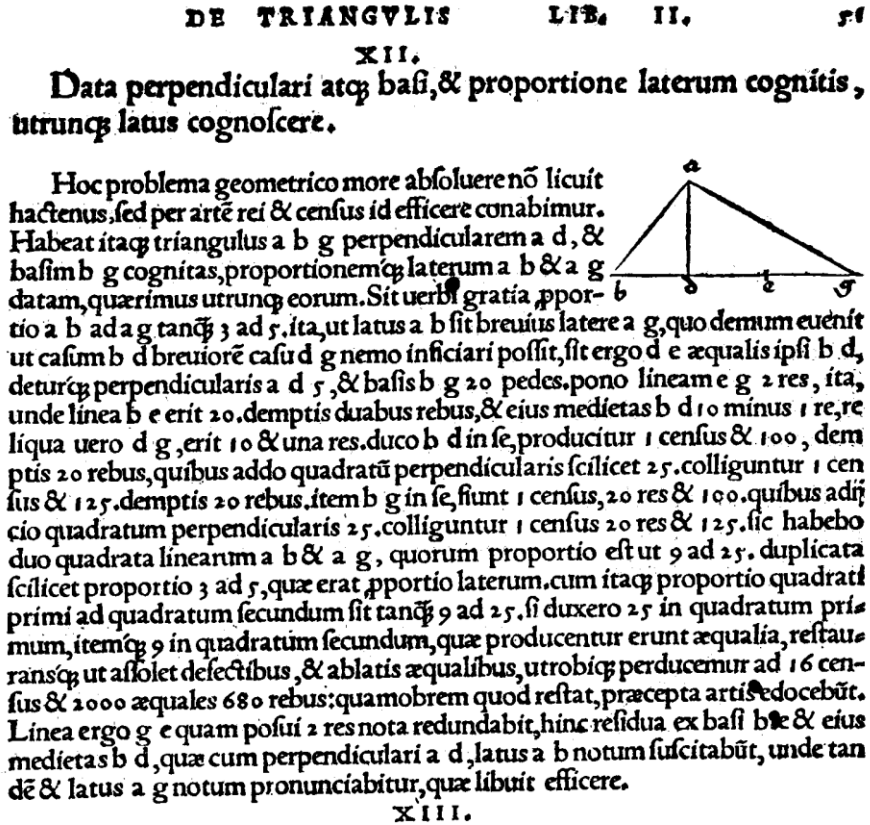
$$16x^2 + 2000 = 680x$$

$$\begin{array}{r} x^2 + 20x + 100 \\ 25 \\ \hline x^2 + 20x + 125 \end{array}$$

Hoc problema geometrico more absolvere non licuit hactenus, sed per arte rei et census id efficere conabimur

"This problem cannot be proven by geometry at this point, but we will endavor to accomplish it by the art of algebra"

Regiomontanus
De Triangulis omnimodis



DE TRIANGVLIS LIB. II. 51

XII.

Data perpendiculari atq; basi, & proportione laterum cognitis, utrunq; latus cognoscere.

Hoc problema geometrico more absoluere nõ licuit hactenus, sed per artẽ rei & census id efficere conabimur. Habeat itaq; triangulus a b g perpendicularem a d, & basim b g cognitas, proportionemq; laterum a b & a g datam, quærimus utrunq; eorum. Sit uerbi gratia, pportio a b ad a g tanq; 3 ad 5. ita, ut latus a b sit breuius latere a g, quo demum euenit ut casum b d breuiorẽ casu d g nemo inficiari possit, sit ergo d e æqualis ipsi b d, deturq; perpendicularis a d 5, & basis b g 20 pedes. pono lineam e g 2 res, ita, unde linea b e erit 20. demptis duabus rebus, & eius medietas b d 10 minus 1 re, reliqua uero d g, erit 10 & una res. duco b d in se, producitur 1 census & 100, demptis 20 rebus, quibus addo quadratũ perpendicularis scilicet 25. colliguntur 1 census & 125. demptis 20 rebus. item b g in se, fiunt 1 census, 20 res & 100. quibus adiicio quadratum perpendicularis 25. colliguntur 1 census 20 res & 125. sic habebo duo quadrata linearum a b & a g, quorum proportio est ut 9 ad 25. duplicata scilicet proportio 3 ad 5, quæ erat pportio laterum. cum itaq; proportio quadrati primi ad quadratum secundum sit tanq; 9 ad 25. si duxero 25 in quadratum primum, itemq; 9 in quadratum secundum, quæ producentur erunt æqualia, restauransq; ut assolet defectibus, & ablatis æqualibus, utrobiq; perducemur ad 16 census & 2000 æquales 680 rebus: quamobrem quod restat, præcepta artis edocebũt. Linea ergo g e quam posui 2 res nota redundabit, hinc residua ex basi b e & eius medietas b d, quæ cum perpendiculari a d, latus a b notum suscitabũt, unde tandẽ & latus a g notum pronunciabitur, quæ libuit efficere.

XIII.

printed edition Nürnberg, 1533 omitting the symbolism

Conclusion

- Question of plagiarism
  - not really meaningful in the context of late 15th cent
  - claims by Franci and Toti Rigatelli are tendentious
- Major contributions of Pacioli
  - restructuring knowledge from abbaco treatises
  - generalizing algebraic solutions in theory (keys)
  - didactical presentation of techniques (second unknown, double false position,..)
  - development of symbolism in the 15th century
  - gateway between manuscripts and print
  - very influencial for 16th cent mathematics